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Rotations with unit timelike quaternions in Minkowski 3-space

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Abstract

With the aid of quaternion algebra, rotation in Euclidean space may be dealt with in a simple manner. In this paper, we show that a unit timelike quaternion represents a rotation in the Minkowski 3-space. Also, we express Lorentzian rotation matrix generated with a timelike quaternion.

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1. Introduction

Quaternions were discovered by Sir William R. Hamilton in 1843 and the theory of quaternions was expanded to include applications such as rotations in the early 20th century. The most important property of the quaternions is that every unit quaternion represents a rotation and this plays a special role in the study of rotations in three-dimensional vector spaces. There are various representations for rotations as orthonormal matrices, Euler angles and unit quaternions in the Euclidean space. But to use the unit quaternions is a

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more useful, natural, and elegant way to perceive rotations compared to other methods. A comparison of these methods can be find in [6,7]. Until the middle of the 20th century, the practical use of quaternions has been minimal in comparison with other methods. But, currently, this situation has changed due to progress in robotics, animation and computer graphics technology [6]. Also, quaternions are an efficient way understanding many aspects of physics and kinematics. Today, quaternions are used especially in the area of computer vision, computer graphics, animation, and to solve optimization problems involving the estimation of rigid body transformations [7].

In this paper, we apply split quaternions to rotations in the Minkowski 3-space. A similar relation to the relationship between quaternions and rotations in the Euclidean space exists between split quaternions and rotations in the Minkowski 3-space. Split quaternions are identified with the semi-Euclidean space $\mathbb{E}^4_2$. Besides, the vector part of split quaternions was identified with the Minkowski 3-space [2]. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector products. We give some properties of the split quaternions in Section 3. But, before this, we remind some concepts of quaternions and the Lorentzian space. In the following sections, we demonstrate how timelike split quaternions are used to perform rotations in the Minkowski 3-space.

2. Preliminary

Quaternion algebra $\mathbb{H}$ is an associative, non-commutative division ring with four basic elements $\{1, i, j, k\}$ satisfying the equalities $i^2 = j^2 = k^2 = -1$ and $i * j = k$, $j * k = i$, $k * i = j$, $j * i = -k$, $k * j = -i$, $i * k = -j$ [10]. Quaternions are a generalization of complex numbers. Also, the quaternion algebra is the even subalgebra of the Clifford algebra of the three-dimensional Euclidean space. The Clifford algebra $C\ell(\mathbb{E}^n_p) = C\ell_{n-p,p}$ for the n-dimensional non-degenerate vector space $\mathbb{E}^n_p$ having an orthonormal base $\{e_1, e_2, \ldots, e_n\}$ with the signature $(p, n-p)$ is the associative algebra generated by 1 and $\{e_i\}$ with satisfying the relations $e_i e_j + e_j e_i = 0$ for $\forall i \neq j$ and $e_i^2 = \begin{cases} -1, \text{ if } i = 1, 2, \ldots, p \\ 1, \text{ if } i = p+1, \ldots, n \end{cases}$. The Clifford algebra $C\ell_{n-p,p}$ has the basis $\{e_{i_1} e_{i_2} \ldots e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$. That is, the division algebra of quaternions $\mathbb{H}$ is isomorphic with the even subalgebra $C\ell^+_{3,0}$ of the Clifford algebra $C\ell_{3,0}$ such that $C\ell^+_{3,0}$ has the basis $\{1, e_2e_3 \to j, e_1e_3 \to k, e_1e_2 \to i\}$ [9].

We write any quaternion in the form $q = (q_1, q_2, q_3, q_4) = q_1 + q_2 i + q_3 j + q_4 k$ or $q = Sq + \vec{V}q$ where the symbols $Sq = q_1$ and $\vec{V}q = q_2 i + q_3 j + q_4 k$ denote the scalar and vector parts of q. If $Sq = 0$ then q is called pure quaternion. The quaternion product $q * q' = (q_1 + q_2 i + q_3 j + q_4 k) * (q'_1 + q'_2 i + q'_3 j + q'_4 k)$ is obtained by distributing the terms on the right as in ordinary algebra, except that the order of the units must be preserved and then replacing each product of units by the quantity given above.

The conjugate of the quaternion q is denoted by Kq, and defined as $Kq = Sq - \vec{V}q$. The norm of a quaternion $q = (q_1, q_2, q_3, q_4)$ is defined by $\sqrt{q * Kq} = \sqrt{Kq * q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$ and is denoted by Nq and we say that $q_0 = q/Nq$ is unit quaternion where $q \neq 0$. The set of unit quaternions is denoted by $\mathbb{H}_1$. Every unit quaternion can be written in the form $q_0 = \cos\theta + \vec{\varepsilon}_0 \sin\theta$ where $\vec{\varepsilon}_0$ is a unit vector satisfying the equality $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = -1$ and is called the axis of the quaternion [6,7,10].

With the aid of the quaternion algebra, rotations in Euclidean space may be dealt with in a simple and elegant manner. If q and r are any non-scalar quaternions, then $r' = qrq^{-1}$ is a quaternion whose norm and scalar part are the same as for r. The vector $\vec{V}r'$ is obtained by revolving $\vec{V}r$ conically about $\vec{V}q$ through twice the angle of q. Thus, if $q = Nq(\cos\theta + \vec{\varepsilon}_0 \sin\theta)$, $\vec{V}r'$ is obtained by revolving $\vec{V}r$ conically about $\vec{\varepsilon}_0$ through the angle 2θ [1,6].

Now, let us give some basic notions of the Lorentzian geometry. The Minkowski 3-space $\mathbb{E}^3_1$ is the Euclidean space $\mathbb{E}^3$ provided with the inner product $\langle \vec{u}, \vec{v} \rangle_L = -u_1v_1 + u_2v_2 + u_3v_3$ where $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a Lorentzian vector $\vec{u}$ in $\mathbb{E}^3_1$ is spacelike, lightlike or timelike if $\langle \vec{u}, \vec{u} \rangle_L > 0$, $\langle \vec{u}, \vec{u} \rangle_L = 0$ or $\langle \vec{u}, \vec{u} \rangle_L < 0$, respectively. The norm of the vector $\vec{u} \in \mathbb{E}^3_1$ is defined by $\|\vec{u}\| = \sqrt{|\langle \vec{u}, \vec{u} \rangle_L|}$. Also, for the timelike vectors in the Minkowski 3-space, we say that a timelike vector is future pointing or past pointing if the first component of the vector is positive or negative, respectively, the Lorentzian vector product $\vec{u} \wedge_L \vec{v}$ of $\vec{u}$ and $\vec{v}$ is defined as follows:

$$\vec{u} \wedge_L \vec{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Moreover, for the vectors $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ in the Minkowski 3-space, the equalities

$$\vec{x} \wedge_L (\vec{y} \wedge_L \vec{z}) = \langle \vec{x}, \vec{y} \rangle_L \vec{z} - \langle \vec{x}, \vec{z} \rangle_L \vec{y} \tag{1}$$

$$\langle \vec{x} \wedge_L \vec{y}, \vec{z} \wedge_L \vec{w} \rangle_L = - \begin{vmatrix} \langle \vec{x}, \vec{z} \rangle_L & \langle \vec{x}, \vec{w} \rangle_L \\ \langle \vec{y}, \vec{z} \rangle_L & \langle \vec{y}, \vec{w} \rangle_L \end{vmatrix} \tag{2}$$

are satisfied. Proof of these identities can be done using vector analysis. The hyperbolic and Lorentzian unit spheres are

$$H^2_0 = \{\vec{a} \in \mathbb{E}^3_1 : \langle \vec{a}, \vec{a} \rangle_L = -1\} \quad \text{and} \quad S^2_1 = \{\vec{a} \in \mathbb{E}^3_1 : \langle \vec{a}, \vec{a} \rangle_L = 1\}$$

respectively. There are two components of H^2_0 passing through $(1, 0, 0)$ and $(-1, 0, 0)$ a future pointing hyperbolic sphere and a past pointing hyperbolic unit sphere, and they are denoted by H^{2+}_0 and H^{2-}_0, respectively.

Theorem 1. *Let $\vec{u}$ and $\vec{v}$ be vectors in the Minkowski 3-space.*

(i) *If $\vec{u}$ and $\vec{v}$ are future pointing (or past pointing) timelike vectors, then $\vec{u} \wedge_L \vec{v}$ is a spacelike vector. $\langle \vec{u}, \vec{v} \rangle_L = -\|\vec{u}\|\|\vec{v}\| \cosh\theta$ and $\|\vec{u} \wedge_L \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sinh\theta$ where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.*

(ii) *If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|\langle \vec{u}, \vec{v} \rangle_L| < \|\vec{u}\|\|\vec{v}\|$, then $\vec{u} \wedge_L \vec{v}$ is timelike, $\langle \vec{u}, \vec{v} \rangle_L = \|\vec{u}\|\|\vec{v}\| \cos\theta$ and $\|\vec{u} \wedge_L \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.*

(iii) *If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|\langle \vec{u}, \vec{v}\rangle_L| > \|\vec{u}\|\|\vec{v}\|$, then $\vec{u} \wedge_L \vec{v}$ is spacelike, $\langle \vec{u}, \vec{v}\rangle_L = -\|\vec{u}\|\|\vec{v}\|\cosh\theta$ and $\|\vec{u} \wedge_L \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sinh\theta$ where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.*
(iv) *If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the equality $|\langle \vec{u}, \vec{v}\rangle_L| = \|\vec{u}\|\|\vec{v}\|$, then $\vec{u} \wedge_L \vec{v}$ is lightlike.*

For further Lorentzian concepts see [3,4,8].

3. Split quaternions

The semi-Euclidean 4-space with 2-index is represented with $\mathbb{E}_2^4$. The inner product of this semi-Euclidean space is

$$\langle \vec{u}, \vec{v}\rangle_{\mathbb{E}_2^4} = -u_1v_1 - u_2v_2 + u_3v_3 + u_4v_4$$

and we say that $\vec{u}$ is timelike, spacelike or lightlike if $\langle \vec{u}, \vec{u}\rangle_{\mathbb{E}_2^4} < 0$, $\langle \vec{u}, \vec{u}\rangle_{\mathbb{E}_2^4} > 0$ and $\langle \vec{u}, \vec{u}\rangle_{\mathbb{E}_2^4} = 0$ for the vector $\vec{u}$ in $\mathbb{E}_2^4$, respectively. Split quaternions $\hat{\mathbb{H}}$ are identified with the semi-Euclidean space $\mathbb{E}_2^4$. Besides, the subspace of $\hat{\mathbb{H}}$ consisting of pure split quaternions $\hat{\mathbb{H}}_0$ is identified with the Minkowski 3-space [2]. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector product.

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, i, j, k\}$ satisfying the equalities $i^2 = -1$, $j^2 = k^2 = 1$ and $i * j = k$, $j * k = -i$, $k * i = j$, $j * i = -k$, $k * j = i$, $i * k = -j$. Also, similar to the division algebra of quaternions, the split quaternion algebra is the even subalgebra of the Clifford algebra of the three-dimensional Lorentzian space. That is, the non-division algebra of split quaternions $\hat{\mathbb{H}}$ is isomorphic with the even subalgebra $C\ell_{2,1}^+$ of the Clifford algebra $C\ell_{2,1}$ where $C\ell_{2,1}^+$ has the basis $\{1, e_2e_3 \to i, e_3e_1 \to k, e_1e_2 \to j\}$ [9].

Scalar and vector parts of split quaternion q are denoted by $Sq = q_1$ and $\vec{V}q = q_2i + q_3j + q_4k$, respectively. The split quaternion product of two quaternions $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined as

$$p * q = p_1q_1 + \langle \vec{V}p, \vec{V}q\rangle_L + p_1\vec{V}q + q_1\vec{V}p + \vec{V}p \wedge_L \vec{V}q$$

where $\langle , \rangle_L$ and $\wedge_L$ are Lorentzian inner product and vector product, respectively. Also, the split quaternion product may be written as

$$p * q = \begin{bmatrix} p_1 & -p_2 & p_3 & p_4 \\ p_2 & p_1 & p_4 & -p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

If $Sq = 0$ then q is called pure split quaternion. Split quaternion product of two pure split quaternions $p = p_2 i + p_3 j + p_4 k$ and $q = q_2 i + q_3 j + q_4 k$ is

$$p * q = \langle \vec{V}p, \vec{V}q \rangle_L + \vec{V}p \times_L \vec{V}q = -p_2 q_2 + p_3 q_3 + p_4 q_4 + \begin{bmatrix} -i & j & k \\ p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \end{bmatrix}. \qquad (3)$$

Let $q = (q_1, q_2, q_3, q_4) = Sq + \vec{V}q$ be a split quaternion. The conjugate of a split quaternion, denoted Kq, is defined as $Kq = Sq - \vec{V}q$. The conjugate of the sum of quaternions is the sum of their conjugates. Since the vector parts of q and Kq differ only in sign, we have $I_q \overset{\text{def}}{=} q * Kq = Kq * q$. Also, for pure split quaternions, since changing the sign of the determinant in (3) is equivalent to interchanging the second and third rows, $K(\vec{V}q * \vec{V}q') = \vec{V}q' * \vec{V}q$. Now, we can define timelike, spacelike and lightlike quaternions, since the set of split quaternions $\hat{\mathbb{H}}$ is identified with semi-Euclidean space $\mathbb{E}^4_2$.

Definition 1. We say that a split quaternion q is spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively, where $I_q = q * Kq = Kq * q$. Obviously, $-I_q = -q_1^2 - q_2^2 + q_3^2 + q_4^2$ is identified with $\langle q, q \rangle_{\mathbb{E}^4_2}$ for the split quaternion $q = (q_1, q_2, q_3, q_4)$.

Definition 2. The norm of $q = (q_1, q_2, q_3, q_4)$ is defined as

$$Nq = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}$$

If $Nq = 1$ then q is called unit split quaternion and $q_0 = q/Nq$ is a unit split quaternion for $Nq \neq 0$. Also, spacelike and timelike quaternions have multiplicative inverses and they hold the property $q * q^{-1} = q^{-1} * q = 1$. And they are constructed by $q^{-1} = \frac{Kq}{Iq}$. Lightlike quaternions have no inverses.

Theorem 2. *Split quaternions satisfy the following properties*

(i) $q * (r * s) = (q * r) * s$,
(ii) $q * (r + s) = q * r + q * s$,
(iii) $K(q * r) = Kr * Kq$,
(iv) $I_{q*r} = I_q I_r$,
(v) $N(q * r) = NqNr$,
(vi) $\vec{V}q$ *is parallel to* $\vec{V}r$ *if and only if* $q * r = r * q$, *for* $\forall q, r, s \in \hat{\mathbb{H}}$.

As a conclusion of this theorem the set of spacelike quaternions is not a group since it is not closed under multiplication. That is, the product of two spacelike quaternions is timelike. Whereas, the set of timelike quaternions denoted by

$$\mathbb{T}\hat{\mathbb{H}} = \{q = (q_1, q_2, q_3, q_4) : q_2, q_3, q_4, q_1 \in R, I_q > 0\}$$

forms a group under the split quaternion product. Also, the set of unit timelike quaternions represented by $\mathbb{T}\hat{\mathbb{H}}_1$ and identified with semi-Euclidean sphere $S_2^3 = \{\vec{a} \in \mathbb{E}_2^4 : \langle \vec{a}, \vec{a} \rangle_{\mathbb{E}_2^4} = 1\}$ is a subgroup of $\mathbb{T}\hat{\mathbb{H}}$.

The vector part of any spacelike quaternion is spacelike since $q_1^2 + q_2^2 - q_3^2 - q_4^2 < 0$ and $0 < q_1^2 < -q_2^2 + q_3^2 + q_4^2 = \langle \vec{V}q, \vec{V}q \rangle_L$. But, vector part of any timelike quaternion can be spacelike, timelike and null. Because of that we examine timelike quaternions whether the vector part is spacelike, timelike or null in $\mathbb{E}_1^3$. This is important especially for polar forms and rotations.

Now, let us express any split quaternion in polar form similar to quaternions and complex numbers. In the rest of this paper, we will examine especially timelike quaternions since the set of timelike quaternions form a group and polar form changes in the case the vector part of timelike quaternion is timelike or spacelike.

(i) Every spacelike quaternion can be written in the form

$$q = Nq(\sinh\theta + \vec{\varepsilon}_0 \cosh\theta)$$

where $\sinh\theta = \frac{q_1}{Nq}$, $\cosh\theta = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{Nq}$ and $\vec{\varepsilon}_0 = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$.

(ii) Every timelike quaternion with spacelike vector part can be written in the form

$$q = Nq(\cosh\theta + \vec{\varepsilon}_0 \sinh\theta)$$

where $\cosh\theta = \frac{q_1}{Nq}$, $\sinh\theta = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{Nq}$, $\vec{\varepsilon}_0 = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$ and $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = 1$.

For example, for the timelike quaternion $q = (2, 1, 0, 2)$, the polar form is $q = \cosh\theta + \vec{\varepsilon}_0 \sinh\theta = 2 + \frac{(1,0,2)}{\sqrt{3}}\sqrt{3}$.

(iii) Every timelike quaternion with timelike vector part can be written in the form

$$q = Nq(\cos\theta + \vec{\varepsilon}_0 \sin\theta)$$

where $\cos\theta = \frac{q_1}{Nq}$, $\sin\theta = \frac{\sqrt{q_2^2-q_3^2-q_4^2}}{Nq}$, $\vec{\varepsilon}_0 = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{q_2^2-q_3^2-q_4^2}}$ is a timelike unit vector in $\mathbb{E}_1^3$ and $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = -1$.

For example, for the timelike quaternion $q = (1, 2, 1, 1)$, the polar form is $q = \sqrt{3}(\cos\theta + \vec{\varepsilon}_0 \sin\theta) = \sqrt{3}\left(\frac{1}{\sqrt{3}} + \frac{(2,1,1)}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{3}}.\right)$.

Considering that a vector in the Lorentzian space are a split quaternion with scalar part is zero, we express following theorems.

Theorem 3. *Every unit timelike quaternion* $q = \cosh\theta + \vec{\varepsilon}_0 \sinh\theta$ *with spacelike vector part can be expressed in the form* $\vec{v} * \vec{u}^{-1}$ *such that* θ *is the hyperbolic angle between Lorentzian vectors* $\vec{u}$ and $\vec{v}$ *satisfying one of the following conditions:*

(i) $\vec{u}$ *and* $\vec{v}$ *are unit timelike vectors which are perpendicular to spacelike unit vector* $\vec{\varepsilon}_0$.
(ii) $\vec{u}$ *and* $\vec{v}$ *are unit spacelike vectors which satisfy the inequality* $|\langle\vec{u}, \vec{v}\rangle_L| > 1$ *and perpendicular to spacelike unit vector* $\vec{\varepsilon}_0$ [5].

For example, the unit timelike quaternion $q = (3, -8, -6, -6)$ with spacelike vector part can be expressed as $\vec{v} * \vec{u}^{-1}$ such that $\vec{u} = (9, 8, 4)$ and $\vec{v} = (3, 2, 2)$ are unit future pointing timelike vectors satisfying the equalities $\cosh\theta = -\langle\vec{u}, \vec{v}\rangle_L = 3$, $\vec{\varepsilon}_0 = \frac{\vec{u}\wedge_L\vec{v}}{\|\vec{u}\wedge_L\vec{v}\|} = \frac{1}{\sqrt{8}}(-8, -6, -6)$ and $\sinh\theta = \sqrt{8}$.

Also, for the unit timelike quaternion $q = (-9, 0, -4, 8)$ with spacelike vector part can be expressed as $\vec{v} * \vec{u}^{-1}$ such that $\vec{u} = (2, 2, 1)$ and $\vec{v} = (-2, 2, 1)$ are unit spacelike vectors satisfying the inequality $|\langle\vec{u}, \vec{v}\rangle_L| > 1$ and the equalities $\vec{\varepsilon}_0 = \frac{\vec{u}\wedge_L\vec{v}}{\|\vec{u}\wedge_L\vec{v}\|} = \frac{1}{\sqrt{80}}(0, -4, 8)$ and $\cosh\theta = -\langle\vec{u}, \vec{v}\rangle_L = 0$.

Theorem 4. *Every unit timelike quaternion* $q = \cos\theta + \vec{\varepsilon}_0 \sin\theta$ *with timelike vector part can be expressed in the form* $\vec{u} * \vec{v}$ *where* $\vec{u}$ *and* $\vec{v}$ *are unit spacelike vectors which are perpendicular to a timelike unit vector* $\vec{\varepsilon}_0$ *and* θ *is the angle between* $\vec{u}$ *and* $\vec{v}$ [5].

For example, the unit timelike quaternion $q = (0, -3, -2, -2)$ with timelike vector part can be expressed as $\vec{u} * \vec{v}$ such that $\vec{u} = (2, 2, 1)$ and $\vec{v} = (2, 1, 2)$ are unit spacelike vectors satisfying the inequality $|\langle\vec{u}, \vec{v}\rangle_L| < 1$ and the equalities $\vec{\varepsilon} = (-3, -2, -2) = \vec{u} \wedge_L \vec{v}$ and $\cos\theta = \langle\vec{u}, \vec{v}\rangle_L = 0$.

One of the corollaries of these theorems is the fact that each great hyperbolical arc of the unit hyperboloid H_0^2 corresponds to a timelike quaternion with spacelike vector part. And using this corollary, we proved sine and cosine laws for hyperbolical triangles on the H_0^{2+} in [5].

4. Rotations with split quaternions in Lorentzian space

There are a lot of methods used to represent rotations like orthonormal matrices, Euler angles and quaternions. Quaternions is the most useful method to represent rotations. If we compare to orthonormal matrices, there are some constraints as each colon of an orthonormal matrix must be unit vector and must be perpendicular to each other. These constraints make it difficult to construct an orthonormal matrix using nine numbers. But, we can construct easily a rotation orthonormal matrix using a unit quaternion. That is, only four numbers are enough to represent a rotation such that there is only one constraint which is that the norm of the quaternion must be equal to 1. This makes it possible to find solutions to some optimization problems involving rotations. Such problems are hard to solve when using orthonormal matrices to represent rotations because of the six non-linear constraints to enforce orthonormality, and the additional constraint det(R) = +1. Every unit quaternion represents a rotation in the Euclidean space. If $\theta = 0$, then this identity rotation is represented by the quaternion $q = (1, 0, 0, 0)$. Also, the rotation of 180°, $\theta = \pi$ about the unit vector $\vec{a}$ (called a flip) represented by the quaternion $q = (0, \vec{a})$. Using a quaternion $q = (q_1, q_2, q_3, q_4)$, we can generate a rotation matrix with

$$R = \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & -2q_1q_4 + 2q_2q_3 & 2q_1q_3 + 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 + 2q_3q_4 & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix} \tag{4}$$

for the given rotation in the Euclidean 3-space. In terms of orthonormal matrices, the rotations about the standard coordinate axes x, y, z through an angle θ are given by

$$R_{q_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad R_{q_y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \text{and}$$

$$R_{q_z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And, we can represent these rotations about the standard coordinate axes with the unit quaternions $q_x = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0, 0\right)$, $q_y = \left(\cos\frac{\theta}{2}, 0, \sin\frac{\theta}{2}, 0\right)$ and $q_z = \left(\cos\frac{\theta}{2}, 0, 0, \sin\frac{\theta}{2}\right)$, respectively.

Each rotation of Euclidean 3-space is represented by a orthogonal rotation matrix with respect to standard basis. These matrices form the three-dimensional special orthogonal group $SO(3)$. Moreover, the function $f : S_3 \simeq \mathbb{H}_1 \to SO(3)$ which sends $q = (q_1, q_2, q_3, q_4)$ to matrix (4) is a homomorphism of groups. The kernel of f is $\{\pm 1\}$ so that the rotation matrix corresponds to the pair $\pm q$ of the unit quaternion. In particular, $SO(3)$ is isomorphic to the quotient group $\mathbb{H}_1 / \{\pm 1\}$ from the first isomorphism theorem.

That is, unit quaternions are very important for representing rotations in the Euclidean 3-space. Is it possible to represent rotations in the Minkowski 3-space with unit timelike quaternions? The answer is yes. Now, let us demonstrate how unit timelike quaternions are used to perform rotations in the Minkowski 3-space and show that every unit timelike quaternion represents a rotation.

Theorem 5. *Let q and r be timelike quaternions. Then, the transformation $R : \mathbb{T}\hat{\mathbb{H}} \to \mathbb{T}\hat{\mathbb{H}}$ defined by $R_q(r) = q * r * q^{-1}$ is a timelike quaternion whose norm and scalar are the same as for r. Also, R_q is linear.*

Proof. The scalar and norm of the $R_q(r)$ are $S(R_q(r)) = S(q * r * q^{-1}) = S(q * q^{-1} * r) = Sr$ and $N(R_q(r)) = Nq * Nr * Nq^{-1} = Nq * Nr * Nq = Nr$. Also, as a conclusion of Theorem 2(iv) the transformation $R_q(r) = q * r * q^{-1}$ is a a timelike quaternion. To see that $R_q(r)$ is linear, let a be real valued scalar and let r and r' be split quaternions, then $R_q(ar + r') = q * (ar + r') * q^{-1} = (q * ar * q^{-1}) + (q * r' * q^{-1}) = a(q * r * q^{-1}) + (q * r' * q^{-1}) = aR_q(r) + R_q(r')$. $\square$

Since scalar part of the timelike quaternion r does not change under the transformation R, we will examine only that how vector part of timelike quaternion $r = (Sr, \vec{V}r)$ changes under the transformation R. Thus, we can interpret that rotation of a vector in the Minkowski 3-space using the split quaternion product $q * \vec{V}r * q^{-1}$. If $q = (q_1, q_2, q_3, q_4)$ is a timelike quaternion, using the transformation law

$$(q * \vec{V}r * q^{-1})_i = \sum_{j=1}^{3} R_{ij}(\vec{V}r)_j$$

the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 & 2q_1q_4 - 2q_2q_3 & -2q_1q_3 - 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 - q_3^2 + q_4^2 & -2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 - 2q_3q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix} \tag{5}$$

where $r = (Sr, \vec{V}r)$. We can see that all rows of this matrix are orthogonal in the Lorentzian mean. In additionally, if we take a unit timelike quaternion $q \in \mathbb{T}\hat{\mathbb{H}}_1$, we obtain an orthogonal rotation matrix in Minkowski 3-space. Each rotation of Minkowski 3-space is represented by a rotation matrix with respect to standard basis. These matrices form the three-dimensional special orthogonal group

$$SO(1,2) = \{R \in M_3(\mathbb{R}) : R^t \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \det R = 1\}.$$

Moreover, the function $\varphi : S_2^3 \simeq \mathbb{T}\hat{\mathbb{H}}_1 \to SO(1,2)$ which sends $q = (q_1, q_2, q_3, q_4)$ to matrix R given in (5) is a homomorphism of groups. The kernel of φ is $\{\pm 1\}$ so that the rotation matrix corresponds to the pair $\pm q$ of the unit quaternion. In particular, $SO(1,2)$ is isomorphic to the quotient group $\mathbb{T}\hat{\mathbb{H}}_1 / \{\pm 1\}$ from the first isomorphism theorem. In another words, for every rotation in the Minkowski 3-space $\mathbb{E}_1^3$, there are two unit timelike quaternions that determine this rotation. These timelike quaternions are q and $-q$. Also, automorphism group of split quaternions $\hat{\mathbb{H}}$ is isomorphic with $SO(1,2)$ [11].

Therefore, a timelike quaternion $q = (q_1, q_2, q_3, q_4)$ is equivalent to a 3×3 orthogonal rotation matrix R_q given by (5). This matrix represents a rotation in the Minkowski 3-space under the condition that $\det R_q = 1$. This is possible with unit timelike quaternions. Also, causal character of vector part of the timelike quaternion q is important. If the vector part of q is timelike or spacelike then the rotation angle is spherical or hyperbolical, respectively. We can see reasons of these cases after the following theorems. Firstly, we give some examples to these conditions.

For example, for the unit timelike quaternion $q=\left(\frac{\sqrt{3}}{2},\frac{1}{2},0,0\right)$ with timelike vector part, the rotation matrix is

$$R_q=\begin{bmatrix}1 & 0 & 0\\ 0 & \dfrac{1}{2} & -\dfrac{\sqrt{3}}{2}\\ 0 & \dfrac{\sqrt{3}}{2} & \dfrac{1}{2}\end{bmatrix}$$

Here, the unit timelike quaternion $q=\left(\frac{\sqrt{3}}{2},\frac{1}{2},0,0\right)$ represents rotation through an angle 120° about the timelike axis $i=(1,0,0)$.

Also, for the unit timelike quaternion $p=(2,1,0,2)$ with spacelike vector part, the rotation matrix is

$$R_p=\begin{bmatrix}9 & 8 & -4\\ 8 & 7 & -4\\ 4 & 4 & -1\end{bmatrix}$$

and p represents a rotation through an hyperbolic angle 2θ about the spacelike axis $\varepsilon=\left(\frac{1}{\sqrt{3}},0,\frac{2}{\sqrt{3}}\right)$ such that $\cosh\theta=2$ and $\sinh\theta=\sqrt{3}$.

Conversely, for a given 3×3 orthonormal rotation matrix in the Minkowski 3-space, we can find the corresponding unit timelike quaternions by using the formulas

$$q_1^2=\frac{1}{4}(1+R_{q1,1}+R_{q2,2}+R_{q3,3}),\qquad q_2=\frac{1}{4q_1}(R_{q3,2}-R_{2,3}),$$
$$q_3=-\frac{1}{4q_1}(R_{q1,3}+R_{q3,1})\qquad q_4=\frac{1}{4q_1}(R_{q2,1}+R_{q1,2})$$

for $q_1\neq 0$. When $q_1=0$, we can find corresponding unit timelike quaternion using the equations $q_3=-\frac{1}{2q_2}R_{q1,2}$, $q_4=-\frac{1}{2q_2}R_{q1,2}$ and $q_2^2=1+q_3^2+q_4^2$. It is enough to determine the timelike quaternion since $0<q_1^2+q_2^2-q_3^2-q_4^2$. When $q_1=0$, we get $0<q_2^2-q_3^2-q_4^2$ or $q_2\neq 0$.

In additionally, for a rotation matrix $R_q\in SO(1,2)$, we can find a unit vector $\vec{\varepsilon}$ defining the axis of rotation R_q is a unit eigenvector for the eigenvalue +1. Then, using the equations $R_{qi,i}$ and $\cosh^2\frac{\theta}{2}-\sinh^2\frac{\theta}{2}=1$ or $\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}=1$, we find the angle θ such that R_q rotates about $\vec{\varepsilon}$ through that angle. Thus, the pair of unit timelike quaternions corresponding to R_q is then $\pm\left(\cos\frac{\theta}{2}+\vec{\varepsilon}\sin\frac{\theta}{2}\right)$ or $\pm\left(\cosh\frac{\theta}{2}+\vec{\varepsilon}\sinh\frac{\theta}{2}\right)$ with respect to axis of rotation $\vec{\varepsilon}$ is timelike or spacelike, respectively.

For example, let us take the rotation matrix $A \in SO(1,2)$.

$$A = \begin{bmatrix} \frac{9}{4} & -2 & \frac{1}{4} \\ -1 & 1 & -1 \\ -\frac{7}{4} & 2 & \frac{1}{4} \end{bmatrix}$$

Eigenvector for the eigenvalue +1 is the rotation axis $\vec{\varepsilon}$. So, we find the rotation axis as $\vec{\varepsilon} = (2, 1, -2)$. Since $\vec{\varepsilon}$ is a spacelike vector, corresponding unit timelike quaternions pair are in the form $\pm(\cosh\frac{\theta}{2} + \vec{\varepsilon}\sinh\frac{\theta}{2})$. Thus, using the equation $A_{1,1} = q_1^2 + q_2^2 + q_3^2 + q_4^2 = \frac{9}{4}$ and $q = \pm\left(\cosh\frac{\theta}{2} + (2, 1, -2)\sinh\frac{\theta}{2}\right)$, we find $q = \pm\left(\frac{3}{\sqrt{8}}, \frac{2}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{-2}{\sqrt{8}}\right)$.

If we take the rotation matrix $B \in SO(1,2)$.

$$B = \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} - 1 & -\frac{\sqrt{2}}{2} - 1 \\ \frac{\sqrt{2}}{2} + 1 & -\frac{1}{2} & -\sqrt{2} - \frac{1}{2} \\ 1 - \frac{\sqrt{2}}{2} & \sqrt{2} - \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

In this case, rotation axis $\vec{\varepsilon} = \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a timelike vector, then corresponding unit timelike quaternions pair for B are in the form $\pm\left(\cos\frac{\theta}{2} + \vec{\varepsilon}\sin\frac{\theta}{2}\right)$. Therefore, using $B_{1,1} = 2$ and $q = \pm\left(\cos\frac{\theta}{2} + \vec{\varepsilon}\sin\frac{\theta}{2}\right)$, we find $\sin\frac{\theta}{2} = \pm\frac{\sqrt{2}}{2}$ and $\cosh\frac{\theta}{2} = \pm\frac{\sqrt{2}}{2}$. That is, the rotation matrix B rotates a vector about the timelike axis $\vec{\varepsilon}$ through 90°.

Theorem 6. *Let $q = \cosh\theta + \vec{\varepsilon}_0\sinh\theta$ be a timelike quaternion with spacelike vector part and $\vec{\varepsilon}$ be a Lorentzian vector. Then the transformation $R_q(\vec{\varepsilon}) = q * \vec{\varepsilon} * q^{-1}$ is a rotation through hyperbolic angle 2θ about the spacelike axis $\vec{\varepsilon}_0$.*

Proof. Firstly, let us choose a dextral set $\{\vec{\varepsilon}_0, \vec{\varepsilon}_1, \vec{\varepsilon}_2\}$ satisfying the equalities $\vec{\varepsilon}_0 \wedge_L \vec{\varepsilon}_1 = \vec{\varepsilon}_2$, $\vec{\varepsilon}_2 \wedge_L \vec{\varepsilon}_0 = -\vec{\varepsilon}_1$, $\vec{\varepsilon}_1 \wedge_L \vec{\varepsilon}_2 = \vec{\varepsilon}_0$, such that $\vec{\varepsilon}_1$ is a timelike vector in the plane of the $\vec{\varepsilon}_0$ and $\vec{\varepsilon}$ with $\langle\vec{\varepsilon}_0, \vec{\varepsilon}_1\rangle_L = 0$. Thus, we can write as $\vec{\varepsilon} = \cosh\tau\vec{\varepsilon}_0 + \sinh\tau\vec{\varepsilon}_1$ or $\vec{\varepsilon} = \sinh\tau\vec{\varepsilon}_0 + \cosh\tau\vec{\varepsilon}_1$ with respect to $\vec{\varepsilon}$ is spacelike or timelike, respectively. Now, to compute $R_q(\vec{\varepsilon}) = q * \vec{\varepsilon} * q^{-1}$, let us find how $\vec{\varepsilon}_0$ and $\vec{\varepsilon}_1$ change under the transformation R_q.

Since $\vec{V}q$ is parallel to $\vec{\varepsilon}_0$, we have $q * \vec{\varepsilon}_0 = \vec{\varepsilon}_0 * q$ and $R_q(\vec{\varepsilon}_0) = q * \vec{\varepsilon}_0 * q^{-1} = \vec{\varepsilon}_0 * q * q^{-1} = \vec{\varepsilon}_0$. Also,

$$\begin{aligned} R_q(\vec{\varepsilon}_1) = q * \vec{\varepsilon}_1 * q^{-1} &= (\cosh\theta + \vec{\varepsilon}_0\sinh\theta) * \vec{\varepsilon}_1 * (\cosh\theta - \vec{\varepsilon}_0\sinh\theta) \\ &= \vec{\varepsilon}_1\cosh^2\theta - \cosh\theta\sinh\theta(\vec{\varepsilon}_1 * \vec{\varepsilon}_0) + \cosh\theta\sinh\theta(\vec{\varepsilon}_0 * \vec{\varepsilon}_1) \\ &\quad -(\vec{\varepsilon}_0 * \vec{\varepsilon}_1) * \vec{\varepsilon}_0\sinh^2\theta \end{aligned}$$

is found. Additionally, we know that $\vec{\varepsilon}_1 * \vec{\varepsilon}_0 = \vec{\varepsilon}_1 \wedge_L \vec{\varepsilon}_0$ for the orthogonal, pure quaternions and $\vec{u} \wedge_L (\vec{v} \wedge_L \vec{w}) = \langle \vec{u}, \vec{v} \rangle_L \vec{w} - \langle \vec{u}, \vec{w} \rangle_L \vec{v}$ is satisfied for the Lorentzian vectors $\vec{u}, \vec{v}, \vec{w}$. Then, since $(\vec{\varepsilon}_0 * \vec{\varepsilon}_1) * \vec{\varepsilon}_0 = (\vec{\varepsilon}_0 \wedge_L \vec{\varepsilon}_1) \wedge_L \vec{\varepsilon}_0 = -\vec{\varepsilon}_1$, we find

$$R_q(\vec{\varepsilon}_1) = \vec{\varepsilon}_1 \cosh 2\theta + \vec{\varepsilon}_2 \sinh 2\theta.$$

It means that $\vec{\varepsilon}$ is rotated through hyperbolic angle 2θ about $\vec{\varepsilon}_0$ by the transformation $R_q(\vec{\varepsilon})$. □

Therefore, a unit timelike quaternion q with spacelike vector part represents a rotation of three-dimensional non-lightlike Lorentzian vector by an angle hyperbolic angle 2θ about the axis of q.

As an example for this theorem, let us take the unit timelike quaternion $q = \cosh\theta + k \sinh\theta$ with spacelike vector part and spacelike vector ε in the plane i and k. Then, $\vec{\varepsilon} = \cosh \tau k + \sinh \tau i$ where τ the hyperbolic angle between $\vec{\varepsilon}$ and k.

Since $\vec{V}q$ is parallel to k, $R_q(k) = q * k * q^{-1} = k$. Also,

$$R_q(i) = q * i * q^{-1} = (\cosh\theta + k \sinh\theta) * i * (\cosh\theta - k \sinh\theta)$$

and using split quaternion product, we obtain

$$R_q(i) = i \cosh 2\theta + j \sinh 2\theta.$$

It means that $R_q(i)$ is a timelike vector obtained by revolving i about k through an hyperbolic angle 2θ in the positive sense. Hence, the spacelike vector $\vec{\varepsilon} = \cosh \tau k + \sinh \tau i$ is transformed into the spacelike vector $R_q(\vec{\varepsilon}) = \cosh \tau k + \sinh \tau R_q(i)$ under the transformation R_q .

In the Minkowski 3-space, the rotations about the standard spacelike coordinate axes $j = (0, 1, 0)$ and $k = (0, 0, 1)$ through the hyperbolic angle θ are represented with the orthonormal matrices

$$R_{q_j} = \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{bmatrix} \quad \text{and} \quad R_{q_k} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or the unit timelike quaternions $q_j = \left(\cosh\frac{\theta}{2}, 0, -\sinh\frac{\theta}{2}, 0\right)$ and $q_k = \left(\cosh\frac{\theta}{2}, 0, 0, \sinh\frac{\theta}{2}\right)$.

Theorem 7. *Let $q = \cos\theta + \vec{\varepsilon}_0 \sin\theta$ be a timelike quaternion with timelike vector part and $\vec{\varepsilon}$ be a Lorentzian vector. Then the transformation $R_q(\vec{\varepsilon}) = q * \vec{\varepsilon} * q^{-1}$ is a rotation through 2θ about the timelike axis $\vec{\varepsilon}_0$.*

Proof. Let us choose a dextral set satisfying the equalities $\vec{\varepsilon}_0 \wedge_L \vec{\varepsilon}_1 = \vec{\varepsilon}_2$, $\vec{\varepsilon}_2 \wedge_L \vec{\varepsilon}_0 = \vec{\varepsilon}_1$, $\vec{\varepsilon}_1 \wedge_L \vec{\varepsilon}_2 = -\vec{\varepsilon}_0$, such that $\vec{\varepsilon}_1$ is a spacelike vector in the plane of the timelike vector $\vec{\varepsilon}_0$

and $\vec{\varepsilon}$ with $\langle\vec{\varepsilon}_0, \vec{\varepsilon}_1\rangle_L = 0$. Thus, we can write as $\vec{\varepsilon} = \cosh\tau\vec{\varepsilon}_0 + \sinh\tau\vec{\varepsilon}_1$ or $\vec{\varepsilon} = \sinh\tau\vec{\varepsilon}_0 + \cosh\tau\vec{\varepsilon}_1$ with respect to $\vec{\varepsilon}$ is timelike or spacelike, respectively. Now, to compute $R_q(\vec{\varepsilon}) = q * \vec{\varepsilon} * q^{-1}$, let us find how $\vec{\varepsilon}_0$ and $\vec{\varepsilon}_1$ change under the transformation R.

Since $\vec{V}q$ is parallel to $\vec{\varepsilon}_0$, we have $q * \vec{\varepsilon}_0 = \vec{\varepsilon}_0 * q$ and $q * \vec{\varepsilon}_0 * q^{-1} = \vec{\varepsilon}_0 * q * q^{-1} = \vec{\varepsilon}_0$. Also, we can find as $R_q(\vec{\varepsilon}_1) = \vec{\varepsilon}_1 \cos 2\theta + \vec{\varepsilon}_2 \sin 2\theta$ using (4) and equalities in the above.

It means that $\vec{\varepsilon}$ is rotated through the angle 2θ about $\vec{\varepsilon}_0$ by the transformation $R_q(\vec{\varepsilon})$. □

Thus, a unit timelike quaternion q with timelike vector part represents a rotation of three-dimensional non-lightlike Lorentzian vector by an angle 2θ about the axis of q.

As an example for this theorem, let us take the unit timelike quaternion $q = \cos\theta + i\sin\theta$ with timelike vector part and unit timelike vector ε in the planes i and j. Then, $\vec{\varepsilon} = \cosh\tau i + \sinh\tau j$ where τ is the hyperbolic angle between $\vec{\varepsilon}$ and i.

Since $\vec{V}q$ is parallel to i, $R_q(i) = q * i * q^{-1} = i$. Also,

$$R_q(j) = q * j * q^{-1} = (\cos\theta + i\sin\theta) * j * (\cos\theta - i\sin\theta)$$

and using split quaternion product, we obtain

$$R_q(j) = j\cos 2\theta + k\sin 2\theta.$$

It means that $R_q(j)$ is a vector obtained by revolving j about i through an angle 2θ in the positive sense. Hence, the timelike vector $\vec{\varepsilon} = \cosh\tau i + \sinh\tau j$ is transformed into the timelike vector $R_q(\vec{\varepsilon}) = \cosh\tau i + \sinh\tau R_q(j)$ under the transformation R_q (see Fig. 1).

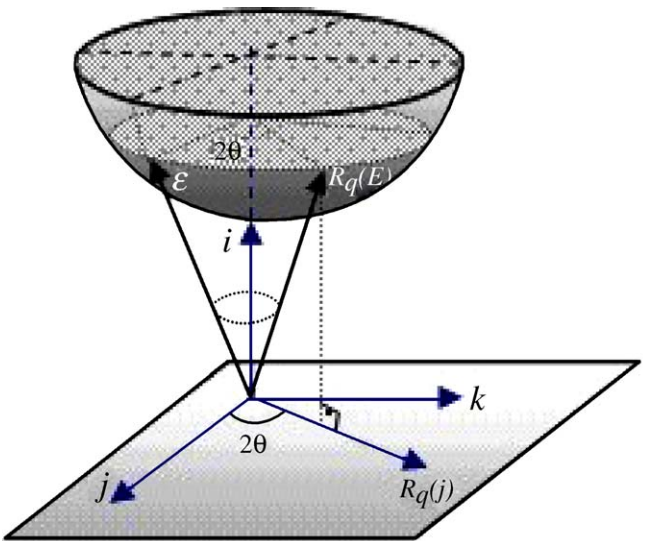

Fig. 1. A unit timelike quaternion q with timelike vector part represents a rotation of three-dimesional non-lightlike Lorentzian vector by an angle 2θ about the axis of q.

The rotation about the standard timelike coordinate axis $i = (1, 0, 0)$ through the angle θ is represented with the orthonormal matrix

$$R_{q_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

or the unit timelike quaternion $q_i = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0, 0\right)$.

5. Some conclusions and remarks

(i) A timelike quaternion rotates a non-lightlike vector to a non-lightlike vector. Even the causal character of the non-lightlike vector is preserved. That is, a timelike vector transforms into a timelike vector and a spacelike vector also transforms into a spacelike vector under the transformation R_q [8].

(ii) All rotations about a timelike axis or a spacelike axis can be expressed with the unit timelike quaternions with timelike vector part or unit timelike quaternions with spacelike vector part, respectively.

(iii) Since $q^{-1}q(\,)q^{-1}q = 1(\,)1$, the inverse of a timelike quaternion q, q^{-1} rotates the same number of degree as q, but the axis points in the opposite direction.

(iv) Since $(-q)^{-1} = -q^{-1}$, the rotations $R_q = q(\,)q^{-1}$ and $R_{(-q)} = (-q)(\,)(-q)^{-1}$ are the same.

(v) For the unit timelike quaternions, the rotation p followed by the rotation q is equivalent to the single rotation $q * p$. Even, while p and q represent rotations the angles α and β about the timelike vectors $\vec{u}$ and $\vec{v}$, respectively, $q * p$ may be represent a rotation through the hyperbolic angle γ about a spacelike vector $\vec{w}$. If the p and q correspond to operators $R_p = p * (\,) * p^{-1}$ and $R_q = q * (\,) * q^{-1}$, the succession of rotations p and q corresponds to the operator $q * p * (\,) * p^{-1} * q^{-1} = (q * p) * (\,) * (q * p)^{-1} = R_{q*p}$.

(vi) If $p = \cos\alpha + u\sin\alpha$ and $q = \cos\beta + u\sin\beta$ are unit timelike quaternions, then the operators R_p and R_q effect rotations of 2α and 2β about timelike vector $\vec{u}$, respectively.

$$\begin{aligned} q * p &= (\cos\beta + u\sin\beta) * (\cos\alpha + u\sin\alpha) \\ &= (\cos\beta\cos\alpha - \sin\alpha\sin\beta) + u(\cos\beta\sin\alpha + \cos\alpha\sin\beta) \\ &= \cos(\alpha+\beta) + u\sinh(\alpha+\beta). \end{aligned}$$

Therefore, the succession of rotations p and q corresponds to the R_{q*p} and effect rotations of $2(\alpha+\beta)$ about $\vec{u}$. That is, the resulting rotation is equivalent to the single rotation $q * p = \cos(\alpha+\beta) + u\sinh(\alpha+\beta)$.

(vii) If $p = \cosh\alpha + u\sinh\alpha$ and $q = \cosh\beta + u\sinh\beta$ are unit timelike quaternions, then the operators R_p and R_q effect rotations of hyperbolic 2α and 2β about spacelike vector $\vec{u}$, respectively.

$$\begin{aligned} q * p &= (\cosh\beta + u\sinh\beta) * (\cosh\alpha + u\sinh\alpha) \\ &= (\cosh\beta\cosh\alpha + \sinh\alpha\sinh\beta) + u(\cosh\beta\sinh\alpha + \cosh\alpha\sinh\beta) \\ &= \cosh(\alpha+\beta) + u\sinh(\alpha+\beta). \end{aligned}$$

So, the succession of rotations p and q corresponds to the R_{q*p} and effect rotations of hyperbolic $2(\alpha+\beta)$ about $\vec{u}$. That is, the resulting rotation is equivalent to the single rotation $q * p = \cos(\alpha+\beta) + u\sinh(\alpha+\beta)$.

(viii) More generally, the succession of rotations $q_1, \ldots, q_n$ is equivalent to the single rotation $q_n * q_{n-1} * \ldots * q_1$ for timelike quaternions.

These conclusions and remarks can be seen also using the corresponding rotation matrices of the timelike quaternions.

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